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The general Kerr–de Sitter metrics in all dimensions

G.W. Gibbons[a], H. Lü[b,1], Don N. Page[b,2], C.N. Pope[b,*,1]

[a] *DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 OWA, UK*
[b] *George P. and Cynthia W. Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, TX 77843-4242, USA*

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Abstract

We give the general Kerr–de Sitter metric in arbitrary space–time dimension $D \geq 4$, with the maximal number $[(D-1)/2]$ of independent rotation parameters. We obtain the metric in Kerr–Schild form, where it is written as the sum of a de Sitter metric plus the square of a null-geodesic vector, and in generalised Boyer–Lindquist coordinates. The Kerr–Schild form is simpler for verifying that the Einstein equations are satisfied, and we have explicitly checked our results for all dimensions $D \leq 11$. We discuss the global structure of the metrics, and obtain formulae for the surface gravities and areas of the event horizons. We also obtain the Euclidean-signature solutions, and we construct complete non-singular compact Einstein spaces on associated S^{D-2} bundles over S^2, infinitely many for each odd $D \geq 5$.

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* Corresponding author. Tel.: +1 979 845 7793; fax: +1 979 845 2590.
E-mail address: pope@physics.tamu.edu (C.N. Pope).
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[2] Present address: Theoretical Physics Institute, 412 Physics Lab., University of Alberta, Edmonton, Alta., Canada T6G 2J1. Research supported in part by the Natural Science and Engineering Research Council of Canada.

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1. Introduction

It is a remarkable fact that there are roughly 10^{20} rotating black holes in the observable universe, and the space–time near each one of them is given to a very good approximation by a simple explicit exact solution of the Einstein vacuum equations called, after its discoverer, the Kerr metric [1]. An almost equally remarkable fact, which played an important role in its derivation, is that, in a certain sense, the metric $g_{\mu\nu}$ is given *exactly* by its linearised approximation around the flat metric $\eta_{\mu\nu}$. More precisely, the metric can be cast into the so-called Kerr–Schild [2] form:

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = \eta_{\mu\nu}\,dx^\mu\,dx^\nu + \frac{2M}{U}(k_\mu\,dx^\mu)^2, \tag{1.1}$$

where k_μ is null and geodesic with respect to both the full metric $g_{\mu\nu}$ and the flat metric $\eta_{\mu\nu}$.

Explicitly, in flat coordinates x, y, z, t, one has

$$k = k_\mu\,dx^\mu = dt + \frac{r(x\,dx + y\,dy) + a(x\,dy - y\,dx)}{r^2 + a^2} + \frac{z\,dz}{r}, \tag{1.2}$$

and

$$U = r + \frac{a^2 z^2}{r^3}, \tag{1.3}$$

where r is defined[3] by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \tag{1.4}$$

Some years later, Myers and Perry [3] were able to show that a simple generalisation of this ansatz yields the exact solution for the metric of a rotating black hole in arbitrary dimensions. Explicitly, they found that, for even space–time dimensions $D = 2n \geq 4$, the null vector is

$$k = k_\mu\,dx^\mu = dt + \sum_{i=1}^{n-1}\frac{r(x_i\,dx_i + y_i\,dy_i) + a_i(x_i\,dy_i - y_i\,dx_i)}{r^2 + a_i^2} + \frac{z\,dz}{r}, \tag{1.5}$$

with

$$U = \frac{1}{r}\left(1 - \sum_{i=1}^{n-1}\frac{a_i^2(x_i^2 + y_i^2)}{(r^2 + a_i^2)^2}\right)\prod_{j=1}^{n-1}(r^2 + a_j^2), \tag{1.6}$$

and

$$\sum_{i=1}^{n-1}\frac{x_i^2 + y_i^2}{r^2 + a_i^2} + \frac{z^2}{r^2} = 1. \tag{1.7}$$

[3] Note that r is *not* the usual radial coordinate in flat space–time.

Substituting (1.5) and (1.6) into (1.1), one obtains the generalisation of the Ricci-flat Kerr metric in $2n$ space–time dimensions, with $(n-1)$ independent rotation parameters a_i in $(n-1)$ orthogonal spatial 2-planes.

If the number of space–time dimensions is odd, say $D = 2n+1$, there are then n pairs of spatial coordinates and no z coordinate, and so the terms involving z are omitted. U is then $1/r$ times the right-hand side of Eq. (1.6).

Shortly after Kerr's paper, Carter discovered [4] a generalisation of the Kerr solution describing a rotating black hole in four-dimensional de Sitter or anti-de Sitter backgrounds, of which he said [5]

> Although I don't think that there is much physical justification for believing a non-zero Λ term, it is perhaps worth quoting the result as a geometrical curiosity.

Times have changed, and there is currently great interest in the cosmological term, both positive and negative, and in various space–time dimensions. Hawking et al. have given [9], without detailed derivation, a generalisation of the five-dimensional Myers–Perry solution to include a cosmological term, and they raised the question of what form it should take in higher dimensions. The purpose of this article is to answer that question.

The solutions of Carter in four dimensions, and Hawking et al. in five dimensions, are already quite complicated, and so one needs some guiding principle to aid one's search. An important clue is provided by Carter's observation that his solution may also be cast in a generalised Kerr–Schild form, in which one replaces the flat Minkowski background metric $\eta_{\mu\nu}$ by that of de Sitter or anti-de Sitter space. In other words, one should now write

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{2M}{U} k_\mu k_\nu, \tag{1.8}$$

where k_μ is null with respect to both the exact metric $g_{\mu\nu}$ and some background metric $\bar{g}_{\mu\nu}$. In our case, $\bar{g}_{\mu\nu}$ will be taken to be the metric of de Sitter or anti-de Sitter space–time. Now quite generally, one may show that if the null vector field k^μ (with index raised using either metric, since both give the same result) is tangent to a null-geodesic congruence,[4] then the Ricci tensor of $g_{\mu\nu}$ is related to that of $\bar{g}_{\mu\nu}$ by [6]

$$R^\mu_\nu = \bar{R}^\mu_\nu - h^\mu_\rho \bar{R}^\rho_\nu + \tfrac{1}{2}\bar{\nabla}_\rho \bar{\nabla}_\nu h^{\mu\rho} + \tfrac{1}{2}\bar{\nabla}^\rho \bar{\nabla}^\mu h_{\nu\rho} - \tfrac{1}{2}\bar{\nabla}^\rho \bar{\nabla}_\rho h^\mu_\nu, \tag{1.9}$$

where $h_{\mu\nu} = (2M/U)k_\mu k_\nu$. (See also [7,8].) Thus, if $\bar{g}_{\mu\nu}$ satisfies $\bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}$, then the full metric $g_{\mu\nu}$ will satisfy the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with the same cosmological constant, provided that $h_{\mu\nu}$ satisfies the *linearised* Einstein equations with respect to the background metric $\bar{g}_{\mu\nu}$. One key reason why the "linearised approximation" is exact in this case is that with $g_{\mu\nu}$ written as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, the fact that k_μ is null implies that the inverse metric is given exactly by $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}$. It is also important to note that (1.9) is written with *mixed* components; one index up and one down.

A key element of Kerr's construction, which reappears in the subsequent work of Carter, Myers and Perry, and Hawking et al., is that the solution is most naturally given in a special coordinate system, related to the well-known Boyer–Lindquist coordinates. These

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[4] A congruence is a family of curves, one and only one passing through every point of space–time. It is null if every tangent vector is null. It is geodesic if every curve is a geodesic.

coordinates have the property that even in the flat-space limit of the Kerr–Myers–Perry solutions, in which the mass M vanishes but the angular velocity parameters a_i of the general solution do not, they do not reduce to standard spherical coordinates. This is clear from the expression (1.7) for the radial coordinate r; the level sets of r are *ellipsoids* rather than *spheres*. In fact they are *ellipsoids of revolution*, meaning that they are invariant under n rotations in the n 2-planes spanned by the coordinates (x_i, y_i). It is convenient therefore to introduce n azimuthal coordinates ϕ_i and n latitudinal coordinates μ_i, such that

$$z_i = x_i + \mathrm{i}y_i = \mu_i\,\mathrm{e}^{\mathrm{i}\phi_i}, \tag{1.10}$$

with $\sum_i \mu_i^2 = 1$. Note that for odd space–time dimensions, in the limit that the rotation parameters a_i vanish, the level sets become spheres, and the coordinates μ_i label the Clifford tori of the odd dimensional spheres S^{2n+1}. The system of ellipsoids also becomes spherical in the special case that all rotation parameters a_i are equal. In fact, this special case is associated with an enhancement of the general $U(1)^n$ symmetry to $U(n)$. It would be interesting to know how these ellipsoidal coordinates are related to other forms of ellipsoidal coordinates that have been introduced in higher dimensional Euclidean space.

An examination of the metrics of Carter and of Hawking et al. also reveals that these special ellipsoidal coordinates can be extended to de Sitter or anti-de Sitter space, and they will play an essential role in what follows.

A conventional static form of the standard (anti)-de Sitter metric in the case of odd space–time dimensions $D = 2n + 1$, with $\bar{R}_{\mu\nu} = (D-1)\lambda\bar{g}_{\mu\nu}$, is

$$\mathrm{d}\bar{s}^2 = -(1-\lambda y^2)\mathrm{d}t^2 + \frac{\mathrm{d}y^2}{1-\lambda y^2} + y^2\sum_{k=1}^{n}(\mathrm{d}\hat{\mu}_k^2 + \hat{\mu}_k^2\,\mathrm{d}\phi_k^2), \tag{1.11}$$

with

$$\sum_{i=1}^{n}\hat{\mu}_i^2 = 1. \tag{1.12}$$

We now define new “spheroidal coordinates” (r, μ_i) according to

$$(1+\lambda a_i^2)y^2\hat{\mu}_i^2 = (r^2 + a_i^2)\mu_i^2, \tag{1.13}$$

where

$$\sum_{i=1}^{n}\mu_i^2 = 1. \tag{1.14}$$

Note that (1.13) and (1.14) imply that

$$y^2 = \sum_{i=1}^{n}\frac{(r^2+a_i^2)\mu_i^2}{1+\lambda a_i^2}. \tag{1.15}$$

In terms of the new coordinates (r, μ_i), the de Sitter metric (1.11) becomes

$$\mathrm{d}\bar{s}^2 = -W(1-\lambda r^2)\mathrm{d}t^2 + F\mathrm{d}r^2 + \sum_{i=1}^{n}\frac{r^2+a_i^2}{1+\lambda a_i^2}(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2)$$

$$+\frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{n}\frac{(r^2+a_i^2)\mu_i\,\mathrm{d}\mu_i}{1+\lambda a_i^2}\right)^2, \tag{1.16}$$

where

$$W\equiv\sum_{i=1}^{n}\frac{\mu_i^2}{1+\lambda a_i^2},\qquad F\equiv\frac{r^2}{1-\lambda r^2}\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}. \tag{1.17}$$

A similar construction can be given in the case that the space–time dimension is even. We shall not present it here, since it will in any case be subsumed by our discussion in the next section.

The de Sitter metrics written in spheroidal coordinates, as in (1.16), form the basis for our construction of the general Kerr–de Sitter metrics in arbitrary dimensions. The constants a_i, which have been introduced in (1.13) merely as parameters in a coordinate transformation of the standard de Sitter metric, acquire the interpretation of genuine rotation parameters for the general Kerr–de Sitter metrics once the square of an appropriate null vector is added, as in (1.8). In D dimensions there are $[(D-1)/2]$ independent rotation parameters a_i, characterising the angular momenta in $[(D-1)/2]$ orthogonal 2-planes.

2. General Kerr–de Sitter metrics in Kerr–Schild form

We begin by presenting our general results for the Kerr–de Sitter metrics, expressed in Kerr–Schild form. To do so, we introduce $n=[D/2]$ coordinates μ_i, which are subject to the constraint

$$\sum_{i=1}^{n}\mu_i^2=1, \tag{2.1}$$

together with $N=[(D-1)/2]$ azimuthal angular coordinates ϕ_i, the radial coordinate r, and the time coordinate t. When the total space–time dimension D is odd, $D=2n+1=2N+1$, there are n azimuthal coordinates ϕ_i, each with period 2π. If D is even, $D=2n=2N+2$, there are only $N=(n-1)$ azimuthal coordinates ϕ_i, which we take to be $(\phi_1,\phi_2,\ldots,\phi_{n-1})$. When D is odd, all the μ_i lie in the interval $0\le\mu_i\le 1$, whereas when D is even, the μ_i all lie in this interval except μ_n, for which $-1\le\mu_n\le 1$.

The Kerr–de Sitter metrics $\mathrm{d}s^2$ satisfy the Einstein equation

$$R_{\mu\nu}=(D-1)\lambda g_{\mu\nu}. \tag{2.2}$$

We first make the definitions

$$W\equiv\sum_{i=1}^{n}\frac{\mu_i^2}{1+\lambda a_i^2},\qquad F\equiv\frac{r^2}{1-\lambda r^2}\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}. \tag{2.3}$$

We find that in $D=2n+1$ dimensions the Kerr–de Sitter metrics are given by

$$\mathrm{d}s^2=\mathrm{d}\bar{s}^2+\frac{2M}{U}(k_\mu\,\mathrm{d}x^\mu)^2, \tag{2.4}$$

where the de Sitter metric $d\bar{s}^2$, the null one-form k_μ, and the function U are given by

$$d\bar{s}^2 = -W(1-\lambda r^2)dt^2 + F\,dr^2 + \sum_{i=1}^{n} \frac{r^2+a_i^2}{1+\lambda a_i^2}(d\mu_i^2 + \mu_i^2\,d\phi_i^2) + \frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{n}\frac{(r^2+a_i^2)\mu_i\,d\mu_i}{1+\lambda a_i^2}\right)^2, \tag{2.5}$$

$$k_\mu\,dx^\mu = W\,dt + F\,dr - \sum_{i=1}^{n}\frac{a_i\mu_i^2}{1+\lambda a_i^2}\,d\phi_i, \tag{2.6}$$

$$U = \sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{n}(r^2+a_j^2). \tag{2.7}$$

Note that the null vector corresponding to the null one-form is

$$k^\mu\partial_\mu = -\frac{1}{1-\lambda r^2}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} - \sum_{i=1}^{n}\frac{a_i}{r^2+a_i^2}\frac{\partial}{\partial\phi_i}. \tag{2.8}$$

In $D = 2n$ dimensions, we find that the Kerr–de Sitter metrics are given by

$$ds^2 = d\bar{s}^2 + \frac{2M}{U}(k_\mu\,dx^\mu)^2, \tag{2.9}$$

where the de Sitter metric $d\bar{s}^2$, the null vector k_μ, and the function U are now given by

$$d\bar{s}^2 = -W(1-\lambda r^2)\,dt^2 + F dr^2 + \sum_{i=1}^{n}\frac{r^2+a_i^2}{1+\lambda a_i^2}d\mu_i^2 + \sum_{i=1}^{n-1}\frac{r^2+a_i^2}{1+\lambda a_i^2}\mu_i^2\,d\phi_i^2 + \frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{n}\frac{(r^2+a_i^2)\mu_i\,d\mu_i}{1+\lambda a_i^2}\right)^2, \tag{2.10}$$

$$k_\mu\,dx^\mu = W\,dt + F\,dr - \sum_{i=1}^{n-1}\frac{a_i\,\mu_i^2}{1+\lambda a_i^2}d\phi_i, \tag{2.11}$$

$$U = r\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{n-1}(r^2+a_j^2). \tag{2.12}$$

In this even dimensional case, where there is no azimuthal coordinate ϕ_n, there is also no associated rotation parameter, and so $a_n = 0$. In this case k^μ is given by

$$k^\mu\partial_\mu = -\frac{1}{1-\lambda r^2}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} - \sum_{i=1}^{n-1}\frac{a_i}{r^2+a_i^2}\frac{\partial}{\partial\phi_i}. \tag{2.13}$$

The vector field k^μ is tangent to a null-geodesic congruence in both even and odd dimensions.

We have been led to the above expressions for the Kerr–Schild forms of the Kerr–de Sitter metrics by casting the previously known $D = 4$ and $D = 5$ Kerr–de Sitter metrics in Kerr–Schild form, and making the natural generalisations that are suggested by symmetries and general considerations of covariance. Checking the correctness of the conjectured forms is a mechanical, if somewhat involved, procedure in any given dimension. We have explicitly verified, by means of computer calculations of (1.9) using *Mathematica*, that the Kerr–de Sitter metrics we have presented here do indeed satisfy the Einstein equations in all dimensions up to and including 11. It is clear from the structure of the metrics that no special features arise that would distinguish the $D \leq 11$ cases from the general case, and thus, we can be confident that our expressions are valid in all dimensions.

3. Kerr–de Sitter metrics in Boyer–Lindquist coordinates

For some purposes it is useful to write the Kerr–de Sitter metrics in Boyer–Lindquist coordinates. Unlike the Kerr–Schild formulation that we used in Section 2, in the Boyer–Lindquist coordinates there are no cross-terms between $\mathrm{d}r$ and the other coordinate differentials. This simplifies the analysis of the event horizons and the causal structure of the metrics.

Let us first consider odd space–time dimensions, $D = 2n + 1$. In Boyer–Lindquist coordinates the Kerr–de Sitter metrics are given by

$$\mathrm{d}s^2 = -W(1-\lambda r^2)\,\mathrm{d}\tau^2 + \frac{U\,\mathrm{d}r^2}{V-2M} + \frac{2M}{U}\left(\mathrm{d}\tau - \sum_{i=1}^{n}\frac{a_i\mu_i^2\,\mathrm{d}\varphi_i}{1+\lambda a_i^2}\right)^2 + \sum_{i=1}^{n}\frac{r^2+a_i^2}{1+\lambda a_i^2}[\mathrm{d}\mu_i^2 + \mu_i^2\,(\mathrm{d}\varphi_i - \lambda a_i\,\mathrm{d}\tau)^2] + \frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{n}\frac{(r^2+a_i^2)\mu_i\,\mathrm{d}\mu_i}{1+\lambda a_i^2}\right)^2, \tag{3.1}$$

where V is defined by

$$V \equiv \frac{1}{r^2}(1-\lambda r^2)\prod_{i=1}^{n}(r^2+a_i^2) = \frac{U}{F}, \tag{3.2}$$

W and F are given in (2.3), and U is given in (2.7). The metric (3.1) is obtained from (2.4) to (2.7) by means of the coordinate transformations

$$\mathrm{d}t = \mathrm{d}\tau + \frac{2M\,\mathrm{d}r}{(1-\lambda r^2)(V-2M)}, \quad \mathrm{d}\phi_i = \mathrm{d}\varphi_i - \lambda a_i\,\mathrm{d}\tau + \frac{2Ma_i\,\mathrm{d}r}{(r^2+a_i^2)(V-2M)}. \tag{3.3}$$

It is useful to note that the sub-determinant for the sector of the metric involving the $x^\alpha = (\tau, \varphi_i)$ directions is given by

$$\det(g_{\alpha\beta}) = -r^2(V - 2M)W \prod_{i=1}^{n} \frac{\mu_i^2}{1 + \lambda a_i^2}. \tag{3.4}$$

In even space–time dimensions, $D = 2n$, the Kerr–de Sitter metrics in Boyer–Lindquist form are given by

$$\begin{aligned} \mathrm{d}s^2 = {} & -W(1 - \lambda r^2)\mathrm{d}\tau^2 + \frac{U\mathrm{d}r^2}{V - 2M} + \frac{2M}{U}\left(\mathrm{d}\tau - \sum_{i=1}^{n-1} \frac{a_i\mu_i^2\mathrm{d}\varphi_i}{1 + \lambda a_i^2}\right)^2 \\ & + \sum_{i=1}^{n} \frac{r^2 + a_i^2}{1 + \lambda a_i^2}\,\mathrm{d}\mu_i^2 + \sum_{i=1}^{n-1} \frac{r^2 + a_i^2}{1 + \lambda a_i^2}\mu_i^2(\mathrm{d}\varphi_i - \lambda a_i\,\mathrm{d}\tau)^2 \\ & + \frac{\lambda}{W(1 - \lambda r^2)}\left(\sum_{i=1}^{n} \frac{(r^2 + a_i^2)\mu_i\,\mathrm{d}\mu_i}{1 + \lambda a_i^2}\right)^2, \end{aligned} \tag{3.5}$$

where V is defined here by

$$V \equiv \frac{1}{r}(1 - \lambda r^2)\prod_{i=1}^{n-1}(r^2 + a_i^2) = \frac{U}{F}, \tag{3.6}$$

W and F are given in (2.3), and U is given in (2.12). As usual in the even dimensional case, there is no rotation associated with the $i = n$ direction, and so $a_n = 0$. The metric (3.5) is obtained from (2.9)–(2.12) by means of the same coordinate transformations (3.3), with the usual understanding that the azimuthal coordinates arise for the index range $1 \le i \le n - 1$ only. The sub-determinant $\det(g_{\alpha\beta})$ is

$$\det(g_{\alpha\beta}) = -r(V - 2M)W \prod_{i=1}^{n-1} \frac{\mu_i^2}{1 + \lambda a_i^2}, \tag{3.7}$$

with V now given by (3.6).

4. Horizons

We consider a metric of generalized Boyer–Lindquist form

$$\mathrm{d}s^2 = X\mathrm{d}\tau^2 + 2Y_i\,\mathrm{d}\tau\,\mathrm{d}\varphi^i + Z_{ij}\,\mathrm{d}\varphi^i\,\mathrm{d}\varphi^j + g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b, \tag{4.1}$$

where the angles φ^i are periodic, with period 2π. The metric g_{ab} and the quantities X, Y_i, Z_{ij} depend only on the x^a coordinates (r, μ_i). The metric will be free of closed timelike curves (CTC's), or almost-CTC's, as long as Z_{ij} is positive definite.

The surfaces r = constant are invariant under time translations, and will be timelike as long as the metric $\mathrm{d}s^2 = X\mathrm{d}\tau^2 + 2Y_i\mathrm{d}\tau\,\mathrm{d}\varphi^i + Z_{ij}\,\mathrm{d}\varphi^i\,\mathrm{d}\varphi^j$ is Lorentzian. To find the intersection of the light-cone with the surfaces, we set $\mathrm{d}\varphi^i = \Omega^i \mathrm{d}\tau$ and $\mathrm{d}\mu_i = 0$, and look for null directions. This leads to an equation for the angular velocities Ω^i:

$$Z_{ij}(\Omega^i + Y^i)(\Omega^j + Y^j) = Z^{ij} Y_i Y_j - X, \tag{4.2}$$

with $Y^i \equiv Z^{ij} Y_j$ and $Z^{ij} Z_{jk} = \delta^i_k$.

Far from the horizon in the stationary region, $\partial/\partial\tau$ is timelike and X is negative. The left-hand side of (4.2) is positive, and there is a cone of possible light-like directions. X may vanish on a timelike ergosurface, but if $Y_i \neq 0$, the right-hand side of (4.2) will remain positive and the surfaces r = constant will remain timelike. As the horizon is approached, the right-hand side of (4.2) approaches zero. The limiting stationary surface r = constant now contains a single null direction, and thus it is a stationary null hypersurface. This is what Carter [5] calls a Killing horizon. The null direction is given by the angular velocities of the horizon $\Omega^i = \Omega^i_{\mathrm{H}}$, where

$$\Omega^i_{\mathrm{H}} = -Y^i|_{r=r_{\mathrm{H}}}. \tag{4.3}$$

A priori, one might imagine that the angular velocities Ω^i_{H} would depend on the latitudinal coordinates μ_i. However, the rigidity property of Killing horizons discussed by Carter [5] implies that they should in fact be *constant* on the horizon. This is indeed the case in our situation. In fact, we find

$$\Omega^i_{\mathrm{H}} = \frac{a_i(1 + \lambda a_i^2)}{r_{\mathrm{H}}^2 + a_i^2}, \tag{4.4}$$

where r_{H} is the radius of the horizon. From (3.4) or (3.7), this occurs at a root of $V - 2M = 0$. It follows that the null generator l of the horizon coincides with an orbit of the Killing vector field

$$l = \frac{\partial}{\partial\tau} + \Omega^i_{\mathrm{H}} \frac{\partial}{\partial\varphi^i}. \tag{4.5}$$

In the Kerr–Schild coordinates of Section 2, one has

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial\tau} + \lambda \sum_i a_i \frac{\partial}{\partial\varphi_i}, \qquad \frac{\partial}{\partial\phi_i} = \frac{\partial}{\partial\varphi_i}, \tag{4.6}$$

and hence

$$l = \frac{\partial}{\partial t} + \sum_i \frac{a_i(1 - \lambda r_{\mathrm{H}}^2)}{r_{\mathrm{H}}^2 + a_i^2} \frac{\partial}{\partial\phi_i}. \tag{4.7}$$

Note that the Kerr–Schild vector field $-k^\mu \partial_\mu$ in Section 2 is future-directed and inward-pointing. In particular, it crosses the horizon, whilst on the horizon the vector field l lies in the horizon. The Kerr–Schild coordinates extend through the future horizon. By contrast, the Boyer–Lindquist coordinates are valid only outside the horizon.

After some calculation, we find that the area of the horizon is given by

$$D = 2n + 1: \qquad A_{\mathrm{H}} = \frac{\mathcal{A}_{2n-1}}{r_{\mathrm{H}}} \prod_{i=1}^{n} \frac{r_{\mathrm{H}}^2 + a_i^2}{1 + \lambda a_i^2}, \tag{4.8}$$

$$D = 2n: \qquad A_{\mathrm{H}} = \mathcal{A}_{2n-2} \prod_{i=1}^{n-1} \frac{r_{\mathrm{H}}^2 + a_i^2}{1 + \lambda a_i^2}, \tag{4.9}$$

where

$$\mathcal{A}_m = \frac{2\pi^{(m+1)/2}}{\Gamma[(m+1)/2]} \tag{4.10}$$

is the volume of the unit m-sphere.

Although l is tangent to the null-geodesic generators of the horizon, it is not affinely parameterised, but rather,

$$l^\mu \nabla_\mu l_\nu = \kappa l_\nu, \tag{4.11}$$

where κ is the surface gravity, which is constant on each connected component of the horizon. Since l is also Killing, we have

$$\tfrac{1}{2} \nabla_\nu L^2 = \kappa l_\nu, \tag{4.12}$$

where

$$-L^2 \equiv l^\mu l_\mu = X + 2Y_i \Omega_{\mathrm{H}}^i + Z_{ij} \Omega_{\mathrm{H}}^i \Omega_{\mathrm{H}}^j. \tag{4.13}$$

Thus, we have

$$\kappa^2 = (\nabla_\mu L)(\nabla^\mu L). \tag{4.14}$$

Note that L vanishes on the horizon, but $\nabla_\mu L$ is non-zero. Near the horizon we find that

$$L^2 \approx U(r_{\mathrm{H}}) \left(\frac{1 - \lambda r_{\mathrm{H}}^2}{V(r_{\mathrm{H}})} \right)^2 V'(r_{\mathrm{H}})(r - r_{\mathrm{H}}), \tag{4.15}$$

and hence we find

$$\kappa = \tfrac{1}{2}(1 - \lambda r_{\mathrm{H}}^2) \frac{V'(r_{\mathrm{H}})}{V(r_{\mathrm{H}})}. \tag{4.16}$$

In odd and even dimensions, this gives

$$D = 2n+1: \qquad \kappa = r_{\mathrm{H}}(1 - \lambda r_{\mathrm{H}}^2) \sum_{i=1}^{n} \frac{1}{r_{\mathrm{H}}^2 + a_i^2} - \frac{1}{r_{\mathrm{H}}}, \tag{4.17}$$

$$D = 2n: \qquad \kappa = r_{\mathrm{H}}(1 - \lambda r_{\mathrm{H}}^2) \sum_{i=1}^{n-1} \frac{1}{r_{\mathrm{H}}^2 + a_i{}^2} - \frac{1 + \lambda r_{\mathrm{H}}^2}{2 r_{\mathrm{H}}}. \tag{4.18}$$

5. Compact Euclidean-Signature Einstein Metrics

In this section, we consider the Euclidean-signature metrics (positive-definite-signature, Riemannian as opposed to pseudo-Riemannian or Lorentz-signature) that are obtained from the Boyer–Lindquist forms (3.1) and (3.5) of the Kerr–de Sitter metrics, by making the replacements

$$\tau = -\mathrm{i}\psi, \qquad a_i = \mathrm{i}\alpha_i. \tag{5.1}$$

Under these transformations the metrics are again real (with real values for ψ and α_i), and now have positive-definite signature.

For general values of the rotation parameters α_i and mass parameter M, the Euclidean-signature metrics do not extend smoothly onto complete, compact, non-singular manifolds. However, this can be achieved for special choices of the parameters, as we shall now show. The discussion follows the one given in [10] for the four-dimensional Euclidean-signature Kerr–de Sitter metric, and in [11] for the five-dimensional Euclidean-signature Kerr–de Sitter metric.

So far, we have presented the Kerr–de Sitter metrics in a local coordinate system, and the ranges of the coordinates r, ψ and φ_i are a priori undetermined. The metrics have various coordinate singularities, at which either components of the metric diverge, or the determinant vanishes. Geometrically, the coordinate singularities arise because the $U(1)^{N+1}$ isometry group generated by $\partial/\partial\varphi^i$ and $\partial/\partial\psi$ has fixed points. In the general case, there is a single linear combination of $\partial/\partial\varphi^i$ and $\partial/\partial\psi$ that vanishes at each fixed-point set. If the manifold is to be smooth and non-singular, and the $U(1)^{N+1}$ isometry group is to act smoothly, then the fixed-point sets must be non-singular embedded sub-manifolds.

From Eq. (3.4) or (3.7), the fixed-point sets occur where any of the μ_i coordinates vanishes, and where $V = 2M$. Near the fixed-point sets, we introduce local coordinates and determine the periods of ψ and φ_i that permit the elimination of the local coordinate singularities. We then must consider the compatibility conditions between these different local coordinate charts. These compatibility conditions give rise to non-trivial constraints on the parameters in the metric.

The simplest metric singularities to consider are those associated with the vanishing of the μ_i coordinates. As in flat space, regularity requires that each coordinate φ_i be periodic with period 2π, keeping the remaining coordinates fixed.

Further coordinate singularities occur where $V - 2M = 0$. We begin by considering one such root, situated say at $r = r_1$. Near $r = r_1$ the metric takes the form

$$\mathrm{d}s^2 \approx \mathrm{d}\rho^2 + \kappa_1^2\rho^2\,\mathrm{d}\psi^2 + g_{ij}\,(\mathrm{d}\varphi^i - \Omega_1^i\,\mathrm{d}\psi)(\mathrm{d}\varphi^j - \Omega_1^j\,\mathrm{d}\psi) + \tilde{g}_{ij}\,\mathrm{d}\mu^i\mathrm{d}\mu^j, \tag{5.2}$$

where ρ is the distance from the horizon and Ω_1^i is now the Euclidean angular velocity, defined to be $-\mathrm{i}\Omega_{\mathrm{H}}^i$ given by (4.4) with $a_i = \mathrm{i}\alpha_i$, namely

$$\Omega_1^i = \frac{\alpha_i(1-\lambda\alpha_i^2)}{r_1^2 - \alpha_i^2}. \tag{5.3}$$

The conical singularity at $\rho = 0$ can be removed by the introduction of local Riemann normal coordinates provided that the period of ψ is chosen to be $2\pi/\kappa_1$ at fixed

$$\varphi_1^i = \varphi^i - \Omega_1^i\psi. \tag{5.4}$$

For future use we perform a further local coordinate transformation to $\psi_1 = \kappa_1\psi$, which has period 2π.

If the cosmological constant is positive, any smooth solution must be compact. If the cosmological constant is negative, then there does not exist any smooth compact solution with continuous isometries. Thus, there are essentially only two possibilities to consider: either λ is positive and we have a compact smooth solution for which the range of r lies between two roots of $V - 2M = 0$, at $r = r_1$ and $r = r_2$, or λ is negative and hence the solution is not compact. We shall consider only the former case.

We have two local charts with coordinates (ψ_1, φ_1^i) and (ψ_2, φ_2^i), all of which have period 2π. The transition function between them is given by the matrix S effecting the linear coordinate transformation

$$\begin{pmatrix} \psi_2 \\ \varphi_2^i \end{pmatrix} = \begin{pmatrix} \dfrac{\kappa_2}{\kappa_1} & 0 \\ \dfrac{\Omega_1^i - \Omega_2^i}{\kappa_1} & \delta^i_j \end{pmatrix} \begin{pmatrix} \psi_1 \\ \varphi_1^j \end{pmatrix}. \tag{5.5}$$

The transition function is a map between two parameterisations of the torus T^{N+1}, where $N = [(D-1)/2]$. It will be well-defined and invertible if and only if the matrix S is invertible, and both it and its inverse have integer entries. In other words, S is an element of $SL(N+1, \mathbb{Z})$. This leads immediately to the requirements

$$|\kappa_1| = |\kappa_2| \equiv \kappa, \tag{5.6}$$

$$\Omega_1^i - \Omega_2^i = \kappa k_i, \tag{5.7}$$

where the k_i are integers. If we take $r_1 < r_2$, then we shall have $\kappa_1 = -\kappa_2 = \kappa > 0$.

The result of this construction is a T^N bundle over S^2. Such bundles are given by a map from the equator of S^2 to T^N, specified by the integers k_i, which characterise the windings of the image of the equator around the N cycles of the torus. The manifold as a whole is an associated S^{D-2} bundle over S^2, with structure group T^N, whose action on S^{D-2} is just rotations around the azimuthal coordinates φ^i. The bundle is trivial or non-trivial according to whether $\sum_i k_i$ is even or odd.

Eq. (5.7) implies

$$\frac{\alpha_i(1-\lambda\alpha_i^2)(r_2^2 - r_1^2)}{\kappa(r_1^2 - \alpha_i^2)(r_2^2 - \alpha_2^2)} = k_i. \tag{5.8}$$

The equality of $|\kappa_1|$ and $|\kappa_2|$ can be achieved by allowing the roots at $r = r_1$ and $r = r_2$ to coincide. (Of course, the distance between the roots in this limit is non-vanishing, since g_{rr} diverges.) It is convenient to write $r_1 = r_0 - \epsilon$ and $r_2 = r_0 + \epsilon$, and take the limit $\epsilon \to 0$. In this limit, the radius $r = r_0$ therefore corresponds to a double root of $V - 2M$. The numerator and denominator of (5.8) both go to zero, with the ratio giving the finite limit

$$k_i = -\frac{8 r_0 V(r_0)\alpha_i(1-\lambda\alpha_i^2)}{(1-\lambda r_0^2)V''(r_0)(r_0^2 - \alpha_i^2)^2}. \tag{5.9}$$

It is convenient to define the dimensionless parameters

$$\nu_i \equiv \frac{\alpha_i}{r_0}, \tag{5.10}$$

and the quantities

$$\alpha \equiv \sum_{i=1}^{N} \frac{1}{1-\nu_i^2} + \sum_{i=1}^{n} \frac{1}{1-\nu_i^2} = D - 1 + 2\sum_{i=1}^{N} \frac{\nu_i^2}{1-\nu_i^2},$$
$$\beta \equiv \frac{\alpha}{\alpha - 2}, \tag{5.11}$$
$$\gamma \equiv 1 + \frac{4\beta}{\alpha^2} \sum_{i=1}^{N} \frac{\nu_i^2}{(1-\nu_i^2)^2},$$

where, as usual, $N = [(D-1)/2]$ is the number of azimuthal coordinates φ^i (and also the number of angular momentum parameters α_i), and $n = [D/2]$ is the number of latitudinal coordinates μ_i, which obey the constraint $\sum_i \mu_i^2 = 1$ (so that only $n-1$ of them are independent), which along with the coordinates r and ψ gives the total space–time dimension $D = N + n + 1$. Note that for odd $D = 2n + 1 = 2N + 1$, we have $n = N$, but for even $D = 2N + 2 = 2n$, we have $n = N + 1$ (and then $\nu_n = 0$).

For r_0 to be a double root of $V - 2M = 0$, we need to set $2M = V(r_0)$ and $V'(r_0) = 0$. From (4.16), the latter condition implies that $\kappa(r_0) = 0$. For all dimensions, odd or even, the conditions for the double root can be solved to give

$$r_0 = (\beta\lambda)^{-1/2}, \qquad M = \alpha^{-1} r_0^{D-3} \prod_{i=1}^{N} (1-\nu_i^2). \tag{5.12}$$

Substituting into (5.9) then gives

$$k_i = \frac{4\nu_i(\beta - \nu_i^2)}{\alpha\gamma(1-\nu_i^2)^2}, \tag{5.13}$$

which are the non-trivial conditions that must be satisfied in order for the metrics to be regular.

In order to obtain the metrics on the S^{D-2} bundles over S^2, we define a new radial coordinate χ by writing

$$r = r_0 - \epsilon \cos\chi \tag{5.14}$$

prior to taking the limit $\epsilon \to 0$. The metric in the $\epsilon \to 0$ limit becomes

$$\begin{aligned} \lambda \mathrm{d}s^2 = {} & \frac{Z}{\alpha\gamma} (\mathrm{d}\chi^2 + \sin^2\chi\, \mathrm{d}\psi_1^2) + \sum_{i=1}^{n} \frac{1-\nu_i^2}{\beta - \nu_i^2} \mathrm{d}\mu_i^2 + \frac{\alpha}{2W} \left(\sum_{i=1}^{n} \frac{1-\nu_i^2}{\beta-\nu_i^2} \mu_i\, \mathrm{d}\mu_i \right)^2 \\ & + \sum_{i=1}^{N} \frac{1-\nu_i^2}{\beta-\nu_i^2} \mu_i^2 \left(\mathrm{d}\varphi_1^i + k_i \sin^2\frac{\chi}{2}\, \mathrm{d}\psi_1 \right)^2 \\ & - \frac{2\beta}{\alpha Z} \left[\sum_{i=1}^{N} \frac{\nu_i \mu_i^2}{\beta - \nu_i^2} \left(\mathrm{d}\varphi_1^i + k_i \sin^2\frac{\chi}{2}\, \mathrm{d}\psi_1 \right) \right]^2, \end{aligned} \tag{5.15}$$

where α, β, and γ are the ν_i-dependent constants given by (5.11), W and Z are functions of the latitudinal coordinates μ_i given by

$$W \equiv \sum_{i=1}^{n} \frac{\beta\mu_i^2}{\beta-\nu_i^2} = 1 + \sum_{i=1}^{N} \frac{\nu_i^2\mu_i^2}{\beta-\nu_i^2}, \tag{5.16}$$

$$Z \equiv \sum_{i=1}^{n} \frac{\mu_i^2}{1-\nu_i^2} = 1 + \sum_{i=1}^{N} \frac{\nu_i^2\mu_i^2}{1-\nu_i^2}, \tag{5.17}$$

and the constants k_i are given by (5.13) (and must be integers for regularity at $\chi = \pi$, thus permitting only discrete choices for the ν_i's).

If the metric (5.15) has positive-definite signature with eigenvalues that are neither zero nor infinite for $0 < \chi < \pi$ and for $0 < \mu_i^2 < 1$, then W, Z, and α must be finite, nonzero, and have the same sign for all allowed values of μ_i. Since one can set any one of the μ_i^2 arbitrarily close to 1 by having the others arbitrarily close to 0, one can see from (5.16) and (5.17) that it is both necessary and sufficient that all the values of $1-\nu_i^2$ and of $\beta-\nu_i^2$ for all i must be non-zero and all have the same sign. We exclude and leave for later investigation the special cases in which one or more of the ν_i^2's are unity.[5] The discussion that follows divides into two cases, according to whether all ν_i satisfy $0 \le \nu_i^2 < 1$, or all satisfy $\nu_i^2 > 1$.

5.1. $0 \le \nu_i^2 < 1$ for all ν_i

If D is even, $D = 2n = 2N+2$, we have $\nu_n = 0$, and thus regularity requires $\nu_i^2 < 1$ for all i. This is also required for D odd, $D = 2n+1$, if one or more of the k_i's (and hence the corresponding ν_i's) is zero. More generally, if $\nu_i^2 < 1$, it follows from (5.11) that $\alpha \ge D-1$, and $1 < \beta \le (D-1)/(D-3) \le 3$. The quantity β attains its maximum value of $(D-1)/(D-3)$ if and only if all $\nu_i = 0$, which leads to the product metric on $S^2 \times S^{D-2}$. Since $\beta > 0$, the second term of the right-hand side of the expression for γ in (5.11) is non-negative, so $1 \le \gamma$, with equality only for the product metric on $S^2 \times S^{D-2}$, for which all $\nu_i = 0$.

We have that

$$\frac{\alpha^2}{\beta} = \alpha(\alpha-2) \ge 4\left(\sum_{i=1}^{N}\frac{1}{1-\nu_i^2}\right)^2 - 4\sum_{i=1}^{N}\frac{1}{1-\nu_i^2}$$
$$> 4\sum_{i=1}^{N}\frac{1}{(1-\nu_i^2)^2} - 4\sum_{i=1}^{N}\frac{1}{1-\nu_i^2} = 4\sum_{i=1}^{N}\frac{\nu_i^2}{(1-\nu_i^2)^2}, \tag{5.18}$$

and therefore from (5.11) we get $1 \le \gamma < 2$.

Since $\beta > 1$, $\beta-\nu_i^2$ and $1-\nu_i^2 > 0$ have the same sign, and it follows that W, Z, and α all have the same sign for all allowed values of the μ_i's. This gives a positive-definite nonsingular metric (5.15) if the regularity condition (5.13) is satisfied.

[5] For example, there is a special case in $D = 5$ with $(k_1, k_2) = (0, 2)$ and $\nu_1 = 1$, which corresponds to the round metric on S^5 [11].

We now show that if $\nu_i^2 < 1$ for all i, the regularity condition (5.13) can be satisfied only if either the metric is a product with $\nu_i = 0$, or if just one k_i is nonzero and takes the value ± 1. For simplicity of presentation, we shall reverse the sign of any coordinate φ_1^i corresponding to a negative ν_i, so that, without loss of generality, we have all ν_i non-negative, and hence all k_i non-negative.

If there is at least one positive ν_i, say of value ν with $0 < \nu < 1$, then (5.11) implies that

$$\alpha \geq D - 3 + \frac{2}{1-\nu^2}. \tag{5.19}$$

Using the fact that $\nu_i^2 < 1$, we have $\beta > 1$ and $\gamma > 1$. This is now a strict inequality, since the sum in the formula (5.11) for γ is positive with at least the one positive ν_i. The regularity condition (5.13) for the k_i corresponding to the positive $\nu_i = \nu$ becomes

$$k_i = \frac{4\nu(1-\nu_i^2/\beta)}{(\alpha-2)\gamma(1-\nu_i^2)^2} < \frac{4\nu}{(\alpha-2)\gamma(1-\nu_i^2)} < \frac{4\nu}{2+(D-5)(1-\nu^2)} < 2 \tag{5.20}$$

for $D \geq 5$. For $D = 4$, it was already proved in [10] that k_i, there called n, must be less than 2. Therefore, when $\nu_i^2 < 1$, all k_i must be either 0 or 1, given that we have chosen coordinates so that all k_i are non-negative.

Next, we prove that there cannot be more than one positive integer k_i, by assuming the contrary and obtaining a contradiction. From (5.13) at fixed α, β, and γ, and using the facts that $0 \leq \nu_i < 1$ and hence $\beta > 1$, it follows that $\mathrm{d}k_i/\mathrm{d}\nu_i > 0$. Thus, for given k_i, there is only one ν_i in the allowed range that solves the regularity condition. By the inequality deduced above, the nonzero k_i's must all equal 1. Therefore, all nonzero ν_i's must take the same value, say ν.

If we assume that there are at least two nonzero ν_i's that have the same positive value ν, we get

$$\alpha \geq D - 5 + \frac{4}{1-\nu^2}, \tag{5.21}$$

and the final two inequalities of (5.20) become

$$k_i < \frac{4\nu}{4+(D-7)(1-\nu^2)} < 1 \tag{5.22}$$

for $D \geq 7$. Thus, the k_i must vanish.

The two remaining cases for which one might have two nonzero ν_i's are $D = 5$ and $D = 6$. Since both of these cases give $N = 2$, only two non-zero ν's are allowed in each of those two cases. These require special arguments to show that $k_i < 1$. These arguments run as follows. For $D = 5$ and $\nu_1 = \nu_2 = \nu$, we get $\alpha = 4/(1-\nu^2)$, $\beta = 2/(1+\nu^2)$, and $\gamma = (1+2\nu^2)/(1+\nu^2)$. Thus, $k_i = \nu(2+\nu^2)/(1+2\nu^2)$, which one can easily show is less than unity for $0 \leq \nu < 1$. For $D = 6$ and $\nu_1 = \nu_2 = \nu$, we get $\alpha = (5-\nu^2)/(1-\nu^2)$, $\beta = (5-\nu^2)/(3+\nu^2)$, and $\gamma = (15+10\nu^2-\nu^4)/(15+2\nu^2-\nu^4)$. Thus, $k_i = 4\nu(5+\nu^2)/(15+10\nu^2-\nu^4)$, which one can also easily show is less than unity for $0 \leq \nu < 1$. Therefore, for all $D \geq 5$, for which more than one nonzero ν_i is allowed by the local solution, we find that one cannot actually have more than one nonzero ν_i for a regular complete compact Einstein metric, assuming $\nu_i^2 < 1$.

To summarize, choosing coordinates such that $k_i \geq 0$ and hence $\nu_i \geq 0$, we find that for $\nu_i < 1$ we can either have all $\nu_i = 0$, or we have one and only one ν_i nonzero, and hence just one non-zero k_i, which must take the value 1. This means that even though our local solution allows up to N nonzero rotation parameters, for complete compact Einstein metrics of Euclidean (positive-definite) signature with any $\nu_i^2 < 1$, there can be only one nonzero rotation parameter, and our solutions reduce to those of [11]. The condition $\nu_i^2 < 1$ is necessarily the case when D is even, and possibly the case when D is odd.

5.2. $\nu_i^2 > 1$ for all ν_i

As discussed previously, if the dimension $D = 2N + 1$ is odd, we can also have regular metrics in which all $\nu_i^2 > 1$. This requires that all the integers k_i be non-zero. In fact, we can show that there are infinitely many inequivalent smooth complete compact Einstein metrics for each odd dimension. In general, there appears to be a unique such metric for each possible choice of the N positive integers k_i, although for $k_1 = k_2 = 1$ in $D = 5$ one has the special case $\nu_1 = \nu_2 = 1$, which is outside the scope of the present discussion. (It leads to a regular compact metric, namely the homogeneous Einstein metric on $T^{1,1}$ [11].)

If all ν_i satisfy $\nu_i^2 > 1$, one has from (5.11) that $\alpha < 0$, and hence $0 < \beta < 1$ and $1 < \gamma$. Using the fact that all the $1 - \nu_i^2$ terms have the same sign, so that the cross terms in the square of the sum contribute positively, we again get from (5.18) that $\gamma < 2$. Thus, in this case the allowed range for γ is $1 < \gamma < 2$. Because in this case $\beta < 1$, a sufficient condition for a positive-definite nonsingular metric (5.15) satisfying the regularity condition (5.13) for odd $D = 2N + 1$ is that both $1 - \nu_i^2$ and $\beta - \nu_i^2$ have the same negative sign for all i.

For this $\nu_i^2 > 1$ case, it is useful to define

$$x_i \equiv \frac{1}{\nu_i^2 - 1}, \qquad A \equiv \sum_i^N x_i, \qquad B \equiv \sum_i^N x_i^2,$$
$$C \equiv \frac{1}{A^2 + 2A + B}, \qquad y_i \equiv A + 1 + x_i. \tag{5.23}$$

Then $\alpha = -2A$, $\beta = A/(A+1)$, $\gamma = 1/[A(A+1)C]$, and the regularity condition (5.13) becomes

$$k_i = 2Cy_i\sqrt{x_i(1+x_i)}. \tag{5.24}$$

Each x_i monotonically decreases from $x_i = \infty$ for $\nu_i = 1$ to $x_i = 0$ for $\nu_i = \infty$, and thus prior to imposing the regularity condition (5.24) that each k_i be a positive integer, each x_i would be allowed to range over the entire positive real axis.

Now define

$$M_{ij} \equiv \frac{\partial \log k_i}{\partial x_j} = D_i\delta_{ij} + a_i + b_j, \tag{5.25}$$

where

$$D_i = \frac{1}{y_i} + \frac{1}{2x_i} + \frac{1}{2(1+x_i)}, \qquad a_i = -2Cy_i, \qquad b_j = \frac{1}{y_j}. \tag{5.26}$$

The Jacobian of the transformation from the x_i to the $\log k_i$ is the determinant of the matrix M_{ij}, which, it appears, is always negative for all positive x_i.[6] Therefore, the map from all positive x_i to $\log k_i$ (and hence to the k_i) is surjective. Thus, if there is any solution for the set of x_i given a set of positive k_i, this solution must be unique.

Extensive numerical searches strongly suggest that for almost all sets of positive integers k_i ($k_1 = k_2 = 1$ for $D = 5$ being the only exception we have found, though as noted above it also leads to a regular compact metric, on $T^{1,1}$ [11]), there is a corresponding unique finite positive solution for the x_i's and hence for the $\nu_i = \sqrt{(1 + x_i)/x_i}$.

It follows that for each set of N entirely positive integers k_i, there is a corresponding smooth complete compact Einstein metric in $D = 2N + 1 \geq 5$ dimensions. This means that there are infinitely many smooth Einstein metrics in all odd dimensions $D \geq 5$. In particular, since the case in which $\sum_i k_i$ is even gives the trivial S^{D-2} bundle over S^2, we have infinitely many smooth Einstein metrics on $S^2 \times S^{D-2}$ for all odd $D \geq 5$.

It is worth remarking that the ν_i are given approximately by

$$\nu_i \approx k_i^{-1} \sum_{j=1}^{N} k_j^2 \tag{5.27}$$

in the case that $\sum_{j=1}^{N} k_j^2 \gg 1$, which applies for all but a small finite number of cases. In particular, this approximation is a good one for any N if at least one of the k_i is large, and it is also a good approximation for arbitrary positive $k_i \geq 1$ if N is large.

An better approximation that is always accurate within 0.026% error is

$$\begin{aligned}\nu_i \approx \frac{1}{ak_i}\Bigg[1 &- \frac{1}{2}a(1 + 5b - 4\lambda_i) - \frac{1}{2}a^2(b + 25b^2 - 24c - 10b\lambda_i + 10\lambda_i^2) \\ &+ \frac{1}{8}a^3(1 + 42b - 43\lambda_i + 167b^2 - 216c - 252b\lambda_i + 304\lambda_i^2 \\ &- 192b^3 + 233bc - 521b^2\lambda_i + 497c\lambda_i + 486b\lambda_i^2 - 529\lambda_i^3)\Bigg],\end{aligned} \tag{5.28}$$

where

$$a \equiv \left(\sum_{i=1}^{N} k_i^2\right)^{-1}, \quad \lambda_i \equiv ak_i^2, \quad b \equiv \sum_{i=1}^{N} \lambda_i^2, \quad c \equiv \sum_{i=1}^{N} \lambda_i^3. \tag{5.29}$$

The terms inside the square brackets that are proportional to unity, a, and a^2 are the first three terms in the power series expansion in a for small a. The terms proportional to a^3 do not form the next term in the power series in a but were chosen to make the error small for cases in which a is not small.

In particular, four linear combinations of the 13 integer coefficients terms inside the round brackets that form the coefficient of $(1/8)a^3$ were determined by requiring that when all the k_i are equal, one gets the first two terms correct (but not the next two) in an expansion in $1/N$ for the coefficient of a^3, that one gets the correct answer of $\nu_1 = \nu_2 = 1$ when $N = 2$

[6] We have verified this explicitly for all $N \leq 5$, and numerically in numerous higher dimensional cases.

and $k_1 = k_2 = 1$, and that one gets nearly the correct answer of $\nu_1 = \nu_2 = \nu_3 = 1 + 2^{1/3}$ when $N = 3$ and $k_1 = k_2 = k_3 = 1$ (as close as possible with integer coefficients inside the round bracket). The remaining 9 of the 13 coefficients inside the round bracket were then chosen to be the integers closest to the values of the coefficients that minimized the sum of the squares of the relative errors of the eighteen different ν_i's for the nine sets of k_i's that have not all k_i's equal in each set and which give $a \geq 0.1$.

This is a somewhat ad hoc procedure for determining the form of Eq. (5.28), but it does lead to an excellent approximation that has a relative error of more than 10^{-4} in only a handful of cases. The worse case is that of $N = 2$, $k_1 = k_2 = 2$, for which the correct answer is $\nu_1 = \nu_2 = \nu = (4 + 10^{1/3} + 10^{2/3})/3 \approx 3.598674508$, whereas Eq. (5.28) gives $\nu_1 = \nu_2 = \nu \approx 1843/512 = 3.599609375 \approx \nu(1 + 0.000259781)$, with a relative error of less than one part in 3800.

One can get exact explicit solutions for the ν_i in some simple cases. One example for each odd $D = 2N + 1 \geq 5$ is the case in which all N values of k_i are equal to k say, and therefore all N values of ν_i are also equal, say to ν. The regularity condition (5.13) then becomes

$$k = \frac{2\nu(N + \nu^2)}{N(N - 1 + 2\nu^2)}. \tag{5.30}$$

If one sets $\nu = Nky$, one gets the cubic equation

$$y^3 - y^2 + \frac{1}{Nk^2}\left(y - \frac{1}{2} + \frac{1}{2N}\right) = 0. \tag{5.31}$$

For large Nk^2, the solution for y is just a bit smaller than unity.

Indeed, solving the cubic explicitly gives

$$\begin{aligned}\nu = \frac{1}{3}Nk \Bigg\{ 1 + \Bigg[1 + \frac{9(N-3)}{4N^2k^2} \\ + 3^{3/2}\left(\frac{N-1}{2N^2k^2} - \frac{13N^2 + 18N - 27}{16N^4k^4} + \frac{1}{N^3k^6}\right)^{1/2}\Bigg]^{1/3} \\ + \Bigg[1 + \frac{9(N-3)}{4N^2k^2} - 3^{3/2}\left(\frac{N-1}{2N^2k^2} - \frac{13N^2 + 18N - 27}{16N^4k^4} + \frac{1}{N^3k^6}\right)^{1/2}\Bigg]^{1/3}\Bigg\},\end{aligned} \tag{5.32}$$

which has the following series expansion in $a = 1/(Nk^2)$:

$$\begin{aligned}\nu = Nk \Big[1 - \tfrac{1}{2}a(1 + N^{-1}) - \tfrac{1}{2}a^2(N^{-1} + N^{-2}) \\ + \tfrac{1}{8}a^3(1 - N^{-1} - 9N^{-2} - 7N^{-3}) + \mathrm{O}(a^4)\Big].\end{aligned} \tag{5.33}$$

The first three terms of this series expansion (5.33) in a coincide with the corresponding ones of (5.28) when all the k_i are equal, but in the $(1/8)a^3$ term, the coefficients of N^{-2} and of N^{-3} in the series expansion above, -9 and -7, respectively, disagree with what (5.28) gives for those coefficients, $+3$ and -26, respectively. The difference is due to the fact that the series expansion (5.33) is that for small a, whereas (5.28) was designed to fit the results for all allowed a to high accuracy.

Special cases of our new compact metrics reduce to previously known examples. When $D = 4$, there is only one parameter ν_i, which is the parameter ν in [10], and (5.13) reduces to Eq. (23) of [10]. In arbitrary dimensions, but with only one ν_i non-zero, the conditions (5.13) reduce to Eq. (3.22) of [11]. In $D = 5$, with both parameters ν_1 and ν_2 non-zero, (5.13) reduces to Eq. (2.22) of [11].

To the best of our knowledge, the metrics (5.15) in arbitrarily high odd dimensions $D = 2N + 1$ and with arbitrarily many different ν_i's are the first explicit compact Euclidean Einstein metrics with arbitrarily high cohomogeneity (cohomogeneity $N - 1 = [(D - 2)/2]$) that are not merely product metrics or warped products with a bounded number of warping parameters.

It is useful to note that the area of the fixed-point set of $\partial/\partial\psi_1$, which occurs at $\chi = 0$, is given by

$$A_{\rm H} = \mathcal{A}_{D-2}(\beta\lambda)^{-(D-2)/2}\prod_{i=1}^{N}\frac{1-\nu_i^2}{1-\nu_i^2/\beta}, \tag{5.34}$$

where $\mathcal{A}_N$ is the volume of the unit N-sphere, given by (4.10). The area of the fixed-point set of $\partial/\partial\psi_2$, which occurs at $\chi = \pi$, is also given by (5.34). The volume of the complete compact metric is given by

$$V_C = \mathcal{A}_D(\beta\lambda)^{-D/2}\frac{2\beta}{\gamma}\prod_{i=1}^{N}\frac{1-\nu_i^2}{1-\nu_i^2/\beta} = \frac{2\mathcal{A}_D}{\gamma\mathcal{A}_{D-2}}\lambda^{-1}A_{\rm H}. \tag{5.35}$$

Note that because $1 \le \gamma < 2$, we have the inequality $\mathcal{A}_D A_{\rm H} < \mathcal{A}_{D-2}\lambda V_C \le 2\mathcal{A}_D A_{\rm H}$.

To summarise, we have seen that complete and non-singular compact Einstein metrics arise as follows. In even dimensions $D \ge 4$, the only regular solutions occur when all k_i except one vanish, and the sole non-vanishing k_i is equal to 1 (or, equivalently modulo trivial coordinate transformations, -1). Correspondingly, all the ν_i vanish except for the one associated with the non-vanishing k_i. The choice of which of the k_i is the non-vanishing one is inessential, since all choices are equivalent up to permutation.

In odd dimensions $D \ge 5$, there is an analogous such solution in which all except one of the k_i vanish, and again all the ν_i except the one associated with the non-zero k_i vanish. In all the above cases, the non-vanishing ν_i satisfies $0 < \nu_i^2 < 1$. However, in the odd dimensional case there are also infinitely many additional regular metrics, in general one for each choice of positive integers k_i. In these cases, all the ν_i satisfy $\nu_i^2 > 1$. (We know of one exceptional case, in $D = 5$, where there is a regular solution with $k_1 = k_2 = 1$, for which $\nu_1 = \nu_2 = 1$; this gives the homogeneous Einstein metric on $T^{1,1}$ [11].)

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Appendices

Here, we present some special cases of the Kerr–de Sitter metrics that we have obtained in this paper. Appendix A describes the Kerr–Schild form of the previously known Kerr–de Sitter metrics [4,9] in $D = 4$ and $D = 5$. Appendices B and C describe Kerr–de Sitter metrics in all odd dimensions $D = 2n + 1$ and even dimensions $D = 2n$, in the special case where the rotation parameters a_i are all set equal. Appendix D describes the special case where all except one of the rotation parameters are set to zero. Under this specialisation, our new metrics reduce to ones that were obtained in arbitrary dimensions in [9]. Finally, Appendix E describes a unified alternative Boyer–Lindquist form of the metrics that applies to both even and odd dimensions and also eliminates one set of off-diagonal terms that appear in the historically-standard Boyer–Lindquist form with a cosmological constant.

Appendix A. Kerr–Schild form of $D = 4$ and $D = 5$ Kerr–de Sitter

Here, we present the explicit Kerr–Schild form of the Kerr–de Sitter metrics in dimensions 4 and 5.

In $D = 4$, we write the coordinates μ_i in (2.9)–(2.12) as $\mu_1 = \sin\theta$, $\mu_2 = \cos\theta$, and we define $\phi_1 = \phi$, $a_1 = a$. The Kerr–de Sitter metric is then written as $ds^2 = d\bar{s}^2 + (2M/U)(k_\mu\, dx^\mu)^2$, where the de Sitter metric $d\bar{s}^2$ is given by

$$d\bar{s}^2 = -\frac{(1-\lambda r^2)\Delta_\theta\, dt^2}{1+\lambda a^2} + \frac{\rho^2\, dr^2}{(1-\lambda r^2)(r^2+a^2)} + \frac{\rho^2\, d\theta^2}{\Delta_\theta} + \frac{(r^2+a^2)\sin^2\theta\, d\phi^2}{1+\lambda a^2}, \tag{A.1}$$

with

$$\rho^2 \equiv r^2 + a^2\cos^2\theta, \qquad \Delta_\theta \equiv 1 + \lambda a^2 \cos^2\theta. \tag{A.2}$$

The null vector k_μ is given by

$$k_\mu\, dx^\mu = \frac{\Delta_\theta\, dt}{1+\lambda a^2} + \frac{\rho^2\, dr}{(1-\lambda r^2)(r^2+a^2)} - \frac{a\sin^2\theta\, d\phi}{1+\lambda a^2}, \tag{A.3}$$

and the function U becomes $U = \rho^2/r$ in this case.

In $D = 5$ we again write the μ_i coordinates in (2.4)–(2.7) as $\mu_1 = \sin\theta$, $\mu_2 = \cos\theta$, and define $\phi_1 = \phi$, $\phi_2 = \psi$, $a_1 = a$ and $a_2 = b$. The Kerr–de Sitter metric is then written as $\mathrm{d}s^2 = \mathrm{d}\bar{s}^2 + (2M/U)(k_\mu\,\mathrm{d}x^\mu)^2$, where the de Sitter metric $\mathrm{d}\bar{s}^2$ is given by

$$\mathrm{d}\bar{s}^2 = -\frac{(1-\lambda r^2)\Delta_\theta\,\mathrm{d}t^2}{(1+\lambda a^2)(1+\lambda b^2)} + \frac{r^2\rho^2\,\mathrm{d}r^2}{(1-\lambda r^2)(r^2+a^2)(r^2+b^2)} + \frac{\rho^2\,\mathrm{d}\theta^2}{\Delta_\theta} + \frac{r^2+a^2}{1+\lambda a^2}\sin^2\theta\,\mathrm{d}\phi^2 + \frac{r^2+b^2}{1+\lambda b^2}\cos^2\theta\,\mathrm{d}\psi^2, \tag{A.4}$$

where we define

$$\rho^2 \equiv r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \qquad \Delta_\theta \equiv 1 + \lambda a^2\cos^2\theta + \lambda b^2\sin^2\theta. \tag{A.5}$$

The null vector k_μ is given by

$$k_\mu\,\mathrm{d}x^\mu = \frac{\Delta_\theta\,\mathrm{d}t}{(1+\lambda a^2)(1+\lambda b^2)} + \frac{r^2\rho^2\,\mathrm{d}r}{(1-\lambda r^2)(r^2+a^2)(r^2+b^2)} - \frac{a\sin^2\theta\,\mathrm{d}\phi}{1+\lambda a^2} - \frac{a\cos^2\theta\,\mathrm{d}\psi}{1+\lambda b^2}. \tag{A.6}$$

The function U is just given by $U = \rho^2$ in this case.

Appendix B. Kerr–de Sitter metrics with $a_i = a$ in $D = 2n + 1$

Here, we consider the case where all the rotation parameters a_i are set equal in the odd dimensional Kerr–de Sitter metrics. From (2.5), this leads to the following special Kerr–de Sitter metric $\mathrm{d}s^2 = \mathrm{d}\bar{s}^2 + (2M/U)(k_\mu\,\mathrm{d}x^\mu)^2$, with

$$\mathrm{d}\bar{s}^2 = -\frac{(1-\lambda r^2)\,\mathrm{d}t^2}{1+\lambda a^2} + \frac{r^2\,\mathrm{d}r^2}{(1-\lambda r^2)(r^2+a^2)} + \frac{r^2+a^2}{1+\lambda a^2}\sum_{i=1}^{n}(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2), \tag{B.1}$$

$$k_\mu\,\mathrm{d}x^\mu = \frac{1}{1+\lambda a^2}\left(\mathrm{d}t - a\sum_{i=1}^{n}\mu_i^2\,\mathrm{d}\phi_i\right) + \frac{r^2\,\mathrm{d}r}{(1-\lambda r^2)(r^2+a^2)}, \tag{B.2}$$

$$U = (r^2+a^2)^{n-1}. \tag{B.3}$$

The solution can be recast in an even simpler form by noting that the terms

$$\sum_{i=1}^{n}(\mathrm{d}\mu_i^2 + \mu_i^2\,\mathrm{d}\phi_i^2) \tag{B.4}$$

in the metric (B.1) describe a unit $(2n-1)$-sphere, which can be viewed as the Hopf fibration over $\mathbb{CP}^{n-1}$, with the metric written as

$$(\mathrm{d}\psi + A)^2 + \mathrm{d}\Sigma_{n-1}^2, \tag{B.5}$$

where ψ has period 2π, $\mathrm{d}\Sigma_{n-1}^2$ is the canonically normalised Fubini-Study metric on $\mathbb{CP}^{n-1}$ (i.e. with $R_{ij} = 2ng_{ij}$), and A is a local potential for the Kähler form $J = (1/2)\mathrm{d}A$ on $\mathbb{CP}^{n-1}$.

It is easily seen that $\sum_i \mu_i^2 \, d\phi_i$ is equal to $(d\psi + A)$, and so we may re-express the de Sitter metric $d\bar{s}^2$ (B.1) and the null vector k_μ (B.2) as

$$d\bar{s}^2 = -\frac{(1-\lambda r^2)\,dt^2}{1+\lambda a^2} + \frac{r^2 \, dr^2}{(1-\lambda r^2)(r^2+a^2)} + \frac{r^2+a^2}{1+\lambda a^2}[(d\psi + A)^2 + d\Sigma_{n-1}^2], \tag{B.6}$$

$$k_\mu \, dx^\mu = \frac{1}{1+\lambda a^2}\,[dt - a\,(d\psi + A)] + \frac{r^2 \, dr}{(1-\lambda r^2)(r^2+a^2)}. \tag{B.7}$$

It is worth remarking that the Fubini-Study metric in (B.6) can be replaced by any Einstein-Kähler metric with the same scalar curvature, and with Kähler form $J = (1/2)dA$, giving us an infinity of further possibilities for generalised Kerr–de Sitter metrics $ds^2 = d\bar{s}^2 + 2M(r^2+a^2)^{1-n}(k_\mu dx^\mu)^2$.

Appendix C. Kerr–de Sitter metrics with $a_i = a$ in $D = 2n$

A similar simplification arises if we set the $(n-1)$ rotation parameters equal in the even dimensional Kerr–de Sitter metrics. Writing $\mu_i = (\mu_\alpha, \mu_n)$, with $1 \le \alpha \le n-1$, and

$$\mu_\alpha = \nu_\alpha \sin\theta, \qquad \mu_n = \cos\theta, \qquad \text{where} \qquad \sum_{\alpha=1}^{n-1} \nu_\alpha^2 = 1, \tag{C.1}$$

we find that upon setting $a_\alpha = a$ in (2.9)–(2.7) the Kerr–de Sitter metric $ds^2 = d\bar{s}^2 + (2M/U)(k_\mu \, dx^\mu)^2$ is given by

$$d\bar{s}^2 = -\frac{(1-\lambda r^2)\Delta_\theta \, dt^2}{1+\lambda a^2} + \frac{\rho^2 \, dr^2}{(1-\lambda r^2)(r^2+a^2)} + \frac{\rho^2 \, d\theta^2}{\Delta_\theta} + \frac{(r^2+a^2)\sin^2\theta}{1+\lambda a^2}\sum_{\alpha=1}^{n-1}(d\nu_\alpha^2 + \nu_\alpha^2 \, d\phi_\alpha^2), \tag{C.2}$$

$$k_\mu \, dx^\mu = \frac{1}{1+\lambda a^2}\left(\Delta_\theta \, dt - a \sin^2\theta \sum_{\alpha=1}^{n-1} \nu_\alpha^2 \, d\phi_\alpha\right) + \frac{\rho^2 \, dr}{(1-\lambda r^2)(r^2+a^2)}, \tag{C.3}$$

$$U = \frac{\rho^2}{r}(r^2+a^2)^{n-2}, \tag{C.4}$$

where we define

$$\rho^2 \equiv r^2 + a^2\cos^2\theta, \qquad \Delta_\theta \equiv 1 + \lambda a^2 \cos^2\theta. \tag{C.5}$$

We can analogously re-express this in terms of a complex projective space (which in this case is $\mathbb{CP}^{n-2}$), giving

$$d\bar{s}^2 = -\frac{(1-\lambda r^2)\Delta_\theta\, dt^2}{1+\lambda a^2} + \frac{\rho^2\, dr^2}{(1-\lambda r^2)(r^2+a^2)} + \frac{\rho^2\, d\theta^2}{\Delta_\theta} + \frac{(r^2+a^2)\sin^2\theta}{1+\lambda a^2}[(d\psi + A)^2 + d\Sigma^2_{n-2}], \tag{C.6}$$

$$k_\mu\, dx^\mu = \frac{1}{1+\lambda a^2}[\Delta_\theta\, dt - a\sin^2\theta(d\psi + A)] + \frac{\rho^2\, dr}{(1-\lambda r^2)(r^2+a^2)}. \tag{C.7}$$

Again, the Fubini-Study metric in (C.6) can be replaced by any Einstein-Kähler metric with the same scalar curvature, and with Kähler form $J = (1/2)dA$, giving us an infinity of further generalised Kerr–de Sitter metrics $ds^2 = d\bar{s}^2 + 2Mr(r^2+a^2)^{2-n}\rho^{-2}(k_\mu\, dx^\mu)^2$.

Appendix D. Kerr–de Sitter metrics with a single rotation parameter

Consider first the case when the space–time dimension is odd, $D = 2n + 1$. When there is a single non-vanishing rotation parameter, we may, without loss of generality, choose it to be a_1. Splitting the i index as $i = (1, \alpha)$, we define

$$\mu_1 = \sin\theta, \quad \mu_\alpha = \nu_\alpha\cos\theta, \quad \text{where} \sum_{\alpha=2}^{n}\nu_\alpha^2 = 1, \qquad a_1 = a, \quad a_\alpha = 0, \quad \phi_1 = \phi. \tag{D.1}$$

It is then straightforward to see that the $D = 2n + 1$ Kerr–de Sitter metrics, given by (2.4)–(2.7), reduce to $ds^2 = d\bar{s}^2 + (2M/U)(k_\mu\, dx^\mu)^2$ with

$$d\bar{s}^2 = -\frac{(1-\lambda r^2)\Delta_\theta\, dt^2}{1+\lambda a^2} + \frac{\rho^2\, dr^2}{(1-\lambda r^2)(r^2+a^2)} + \frac{\rho^2}{\Delta_\theta}\, d\theta^2 + \frac{(r^2+a^2)\sin^2\theta\, d\phi^2}{1+\lambda a^2} + r^2\cos^2\theta\, d\Omega^2_{2n-3}, \tag{D.2}$$

$$k_\mu\, dx^\mu = \frac{\Delta_\theta\, dt}{1+\lambda a^2} - \frac{\rho^2\, dr}{(1-\lambda r^2)(r^2+a^2)} - \frac{a\sin^2\theta\, d\phi}{1+\lambda a^2}, \tag{D.3}$$

$$U = \rho^2 r^{2n-2}, \tag{D.4}$$

where $d\Omega^2_{2n-3} = \sum_\alpha(d\nu_\alpha^2 + \nu_\alpha^2\, d\phi_\alpha^2)$ is the unit metric on S^{2n-3}, and

$$\rho^2 \equiv r^2 + a^2\cos^2\theta, \qquad \Delta_\theta \equiv 1 + \lambda a^2\cos^2\theta. \tag{D.5}$$

The metric ds^2 is equivalent to the single-rotation Kerr–de Sitter metric given in [9], rewritten in Kerr–Schild coordinates.

The discussion for the even dimensional case, $D = 2n$, is almost identical. Specialising (2.9)–(2.12) by setting $a_1 = a$, $a_\alpha = 0$ for $2 \le \alpha \le n - 1$, we find that the single-parameter Kerr–de Sitter metric takes the same form as in the odd dimensional case above, with k_μ

given by (D.3), except that now we have

$$d\bar{s}^2 = -\frac{(1-\lambda r^2)\Delta_\theta\, dt^2}{1+\lambda a^2} + \frac{\rho^2\, dr^2}{(1-\lambda r^2)(r^2+a^2)} + \frac{\rho^2\, d\theta^2}{\Delta_\theta} + \frac{(r^2+a^2)\sin^2\theta\, d\phi^2}{1+\lambda a^2} + r^2\cos^2\theta\, d\Omega_{2n-4}^2, \tag{D.6}$$

$$U = \rho^2 r^{2n-3}. \tag{D.7}$$

Again, this is equivalent to the single-rotation Kerr–de Sitter metric given in [9], rewritten in Kerr–Schild coordinates.

Appendix E. Unified alternative Boyer–Lindquist form of the metrics

Here we give a unified form for the Kerr–de Sitter metrics in all dimensions, even or odd. Furthermore, we present it in a somewhat simpler form than the metric has in the "standard" Boyer–Lindquist coordinates of (3.1) and (3.5).

We begin by defining the "evenness" integer

$$\epsilon \equiv n - N \equiv [D/2] - [(D-1)/2] = (D-1) \bmod 2, \tag{E.1}$$

which is 1 for even D and 0 for odd D. The number of latitudinal coordinates μ_i is then $n = N + \epsilon$, where N is the number of azimuthal angular coordinates ϕ_i, φ_i, or (in this Appendix) $\hat{\varphi}_i$. Since the latitudinal coordinates obey the one constraint (2.1), $\sum_i^{N+\epsilon}\mu_i^2 = 1$, there are actually $N+\epsilon-1$ independent latitudinal coordinates. This is the same as the number N of azimuthal angular coordinates in even dimensions $D = 2N+2 = 2n$, but one less in odd dimensions $D = 2N+1 = 2n+1$. In all cases $D = 2N+\epsilon+1$, which is the sum of the single time coordinate t or τ, the one radial coordinate r, the $N+\epsilon-1$ independent latitudinal coordinates μ_i, and the N azimuthal angular coordinates ϕ_i, φ_i, or $\hat{\varphi}_i$.

In this Appendix, instead of the second coordinate transformation of (3.3), we shall use

$$d\phi_i = d\hat{\varphi}_i + \frac{2Ma_i\, dr}{(r^2+a_i^2)(V-2M)}, \tag{E.2}$$

so $\hat{\varphi}_i = \varphi_i - \lambda a_i\, d\tau$ in terms of the φ_i used in the alternative Boyer–Lindquist coordinates of Section 3. Then the Kerr–de Sitter metrics become, in both odd and even dimensions,

$$ds^2 = -W(1-\lambda r^2)\, d\tau^2 + \frac{2M}{VF}\left(W\, d\tau - \sum_{i=1}^{N}\frac{a_i\mu_i^2\, d\hat{\varphi}_i}{1+\lambda a_i^2}\right)^2 + \sum_{i=1}^{N}\frac{r^2+a_i^2}{1+\lambda a_i^2}\mu_i^2 d\hat{\varphi}_i^2 + \frac{VF dr^2}{V-2M} + \sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{1+\lambda a_i^2}d\mu_i^2 + \frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{1+\lambda a_i^2}\mu_i d\mu_i\right)^2, \tag{E.3}$$

where

$$W \equiv \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{1+\lambda a_i^2}, \tag{E.4}$$

$$V \equiv r^{\epsilon-2}(1-\lambda r^2)\prod_{i=1}^{N}(r^2+a_i^2), \qquad F \equiv \frac{1}{1-\lambda r^2}\sum_{i=1}^{N+\epsilon}\frac{r^2\mu_i^2}{r^2+a_i^2}. \tag{E.5}$$

In this alternative Boyer–Lindquist form of the Kerr–de Sitter metric, the null 1-form is

$$k_\mu \, \mathrm{d}x^\mu = \frac{VF\,\mathrm{d}r}{V-2M} + W\,\mathrm{d}\tau - \sum_{i=1}^{N}\frac{a_i\,\mu_i^2\,\mathrm{d}\hat{\varphi}_i}{1+\lambda\,a_i^2}, \tag{E.6}$$

and the corresponding null vector is

$$k^\mu \partial_\mu = \frac{\partial}{\partial r} - \frac{V}{V-2M}\left(\frac{1}{1-\lambda r^2}\frac{\partial}{\partial \tau} + \sum_{i=1}^{N}\frac{a_i}{r^2+a_i^2}\frac{\partial}{\partial\hat{\varphi}_i}\right). \tag{E.7}$$

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